Beam propagation in uniaxial anisotropic media

J. A. Fleck, Jr., and M. D. Feit

Lawrence Livermore National Laboratory, University of California, Livermore, California 94550

Received January 29, 1983

Paraxial wave equations are derived for the propagation of beams in uniform uniaxial anisotropic media. The equations are generalized to the case of nonuniform media with weakly varying refractive indices. An ordinary wave beam is governed by a standard paraxial equation, whereas an extraordinary wave beam is governed by a paraxial wave equation, which involves both a displacement relative to the position of an ordinary wave beam and a rescaling of one transverse coordinate. The solution to the latter equation for a propagating Gaussian beam displays a distortion of both shape and phase front. Numerical results for diffraction by a uniformly illuminated circular aperture in a calcite medium display various anomalies ascribable to a loss of circular symmetry.

1. INTRODUCTION

The theory of plane-wave propagation in anisotropic media, which goes back to the work of Fresnel in the past century, is outlined in most standard texts on optics. This theory is applicable to many situations, but it is inadequate for treating beams, which are composed of many plane waves propagating in different directions at the same time. The theory of beam propagation in isotropic media, which was given considerable impetus by the invention of the laser, is well developed, but scant attention has been paid to propagation in anisotropic media. The subject is of fundamental interest because one should be able to understand, for example, how diffraction behaves in anisotropic as well as in isotropic media, and it is of practical interest because it applies to the fields in waveguide devices used in integrated optics that are either fabricated from birefringent media, such as LiNbO₃, or rely on electrically induced birefringence. Although the fields in slab waveguides fabricated from anisotropic media can be expressed as simple combinations of standing and plane waves, the fields in waveguides with graded-index profiles or in two-dimensional (2D) waveguides must be described as general superpositions of plane waves or, equivalently, as beams.

Beam-propagation theory is based on the assumption of a plane-polarized beam, which in turn entails the assumption of paraxiality, although in certain cases nonparaxiality may be taken into account approximately. In any case, the above assumptions lead to the well-known scalar paraxial wave equation, which is formally equivalent to the Schrödinger equation and which has had too many applications to review here.

In this paper we derive a set of modified paraxial wave equations that describe propagation in a uniaxial anisotropic medium and present solutions to illustrate the differences between beam propagation in isotropic and anisotropic media. The equations are derived under the assumption of uniform anisotropy, but they are readily generalizable to the case of small spatial variations in refractive index.

The paper is organized as follows: In Section 2 we derive a set of generalized wave equations that govern propagation in uniform anisotropic media. These equations take the place of the vector Helmholtz equation that governs the propagation of fields in uniform isotropic media and permit a unified description of both plane-wave and beam propagation in uniaxial media. In Section 3 we derive from these generalized wave equations some well-known results applicable to plane-wave propagation in uniaxial media. In Section 4 we derive paraxial wave equations from the generalized wave equations. A standard paraxial wave equation governs the propagation of the ordinary wave beam. A rotated coordinate system facilitates the derivation of the paraxial wave equation for the extraordinary wave beam, which differs in some important respects from the standard paraxial wave equation. In Section 5 we outline the solution of the paraxial wave equation for an extraordinary wave beam in terms of Fourier transforms. Results are presented for Gaussian-beam propagation and diffraction by a uniformly illuminated circular aperture. The extension of the various wave equations to media with weakly guiding refractive-index variations is discussed in Section 6. Section 7 contains a summary and conclusions, and the Appendix deals with the displacement of an extraordinary wave beam from the direction of the wave vector.

2. BASIC EQUATIONS

We begin by writing Maxwell’s equations for a nonmagnetic medium:

\[ \nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = 0, \]  
\[ \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} = 0, \]  
\[ \nabla \cdot \mathbf{D} = 0. \]

On elimination of \( \mathbf{H} \) from Eqs. (1a) and (1b), one obtains

\[ \nabla \times \nabla \times \mathbf{E} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{D}}{\partial t^2}. \]

For a single-frequency component with circular frequency \( \omega \), Eq. (2) can be written as

\[ \nabla^2 \mathbf{E} - \nabla (\nabla \cdot \mathbf{E}) + \frac{\omega^2}{c^2} \mathbf{D} = 0. \]

For a uniform isotropic medium, the constitutive relation
\[ D = \varepsilon E \]  
(4)

and Eq. (1c) imply that
\[ \nabla \cdot E = 0. \]  
(5)

The field is therefore transverse, and Eq. (3) reduces to the vector Helmholtz equation
\[ \nabla^2 E = \frac{\omega^2 c}{c^2} E = 0. \]  
(6)

For a uniform anisotropic medium, on the other hand, the constitutive relation becomes
\[ D = \varepsilon \cdot E, \]  
(7)

where \( \varepsilon \) is a dyadic expressible in diagonal matrix form as
\[ \varepsilon = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}. \]  
(8)

The essential change is, of course, that Eq. (1c) no longer implies that \( \nabla \cdot E = 0 \), and the electric field cannot be transverse to the direction of wave propagation.

Let us assume that \( \varepsilon_x = \varepsilon_y \neq \varepsilon_z \) and that \( \varepsilon_x \) and \( \varepsilon_z \) do not vary with position. Thus the \( z \) axis becomes the optic axis. Writing out Eq. (1c), we have
\[ \varepsilon_x \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} \right) + \varepsilon_z \frac{\partial E_z}{\partial z} = 0. \]  
(9)

Consequently
\[ \nabla \cdot E = (1 - \gamma^2) \frac{\partial E_z}{\partial z}, \]  
(10)

where
\[ \gamma^2 = \frac{\varepsilon_z}{\varepsilon_x} = \frac{n_x^2}{n_z^2} \]  
(11)

and where \( n_x \) and \( n_z \) are the extraordinary and ordinary refractive indices, respectively. We now write out Eq. (3) for each component:
\[ \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - (1 - \gamma^2) \frac{\partial^2 E_x}{\partial x \partial y} + \frac{\omega^2}{c^2} \varepsilon_x E_x = 0, \]  
(12a)

\[ \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} - (1 - \gamma^2) \frac{\partial^2 E_y}{\partial x \partial z} + \frac{\omega^2}{c^2} \varepsilon_y E_y = 0, \]  
(12b)

\[ \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \gamma^2 \frac{\partial^2 E_z}{\partial z^2} + \frac{\omega^2}{c^2} \varepsilon_z E_z = 0. \]  
(12c)

These are the basic equations that replace the vector Helmholtz equation (6) in a uniaxial medium.

### 3. PLANE-WAVE ANALYSIS

It is instructive to derive plane-wave solutions for Eqs. (12). Even though the results are well known, the derivation will be a variant on the usual one, and the results will, in any case, be needed for the derivation of the equations for beam propagation.

We look for solutions of the form
\[ E = E_0 \exp[-i(k_x x + k_y y + k_z z)]. \]  
(13)

Substitution of expression (13) into Eqs. (12) gives
\[ -(k_x^2 + k_y^2 + k_z^2) + k_0^2 n_z^2 E_z = 0, \]  
(14a)

\[ -(k_x^2 + k_y^2 + k_z^2) + k_0^2 n_x^2 E_x = 0, \]  
(14b)

\[ -(k_x^2 + k_y^2 + k_z^2) + k_0^2 n_y^2 E_y = 0, \]  
(14c)

where \( k_0 = \omega/c \).

Equations (14) permit a nontrivial solution for \( E_x^0, E_y^0 \), and \( E_z^0 \) provided that the determinant of the coefficients vanishes, or
\[ \left[ -(k_x^2 + k_y^2 + k_z^2) + k_0^2 n_x^2 \right] \left[ -(k_x^2 + k_y^2 + k_z^2) + k_0^2 n_y^2 \right] \left[ -(k_x^2 + k_y^2 + k_z^2) + k_0^2 n_z^2 \right] = 0. \]  
(15)

We shall refer to the two possible characteristic wave solutions as the ordinary and the extraordinary waves, respectively.

The ordinary-wave characteristic solution is determined by
\[ k_x^2 + k_y^2 + k_z^2 = n_o^2 k_0^2, \]  
(16a)

\[ E_x^0 = 0, \]  
(16b)

\[ (E_x^0)^2 + (E_y^0)^2 \neq 0. \]  
(16c)

The relationship between \( E_x^0 \) and \( E_z^0 \) must be determined from Eqs. (5), (13), and (16b) and is
\[ E_y^0 = -\frac{k_x}{k_y} E_z^0. \]  
(17)

In the absence of a \( z \) component the electric field is perpendicular to both the \( z \) axis and the direction of wave propagation, i.e., the wave is polarized perpendicular to the principal plane formed by these two directions. Finally, the phase velocity, which is independent of direction, is, from Eq. (16a),
\[ v_p = c/n_o. \]  
(18)

The extraordinary-wave characteristic solution is determined by the second solution to Eq. (15), or
\[ k_x^2 + k_y^2 + (n_o/n_x)k_z^2 = k_0^2 n_x^2, \]  
(19a)

\[ E_x^0 \neq 0, \]  
(19b)

\[ E_y^0 = -\frac{(1 - n_o^2/n_x^2)k_x k_z E_z^0}{k_0^2 n_o^2 - (k_x^2 + k_y^2 + k_z^2)}, \]  
(19c)

\[ E_z^0 = -\frac{(1 - n_o^2/n_x^2)k_y k_z E_z^0}{k_0^2 n_o^2 - (k_x^2 + k_y^2 + k_z^2)}. \]  
(19d)

The two field vectors specified by Eqs. (16b)–(17) and Eqs. (19b)–(19d), respectively, are clearly orthogonal. Hence the extraordinary wave is polarized in the principal plane. Let us write
\[ k_x^2 + k_y^2 + k_z^2 = n_z^2 k_0^2, \]  
(20)

where \( v_p = c/n \) represents the phase velocity of the extraordinary wave. Putting
\[ (k_x^2 + k_y^2)/n_z^2 k_0^2 = \sin^2 \theta, \]  
(21a)

\[ k_z^2/n_z^2 k_0^2 = \cos^2 \theta, \]  
(21b)

we can reduce Eq. (19a) to
which is equivalent to
\[ v_p^2 = v_o^2 \sin^2 \theta + v_o^2 \cos^2 \theta, \tag{23} \]
where \( v_o = c/n_o \) and \( v_e = c/n_e \) are the extraordinary and ordinary phase velocities, respectively.

In the case of \( \theta = 0 \), one can conclude that \( k_x = k_y = 0 \) from Eq. (21a). Hence, from Eqs. (19c) and (19d), \( \varepsilon_x = \varepsilon_y = 0 \). From Eq. (9), however, \( \varepsilon_z = 0 \) as well. Hence the extraordinary-wave characteristic solution cannot exist, and only the ordinary-wave solution is possible.

4. DERIVATION OF WAVE EQUATIONS FOR PARAXIAL BEAMS

With no loss of generality we can assume that a beam is propagating in a direction fixed in the \( y-z \) plane (see Fig. 1) and making an angle \( \theta \) with the \( z \) axis. It will be convenient to introduce a new set of coordinate axes obtained by a rotation through an angle \( \pi/2 - \theta \) about the \( x \) axis. The new coordinates are given by
\[ x' = x, \quad y' = y \sin \theta + z \cos \theta, \quad z' = -y \cos \theta + z \sin \theta, \tag{24} \]
and derivatives in the unrotated coordinate system are related to the derivatives in the rotated coordinate system through
\[ \frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial y'} - \cos \theta \frac{\partial}{\partial z'}, \quad \frac{\partial}{\partial z} = \cos \theta \frac{\partial}{\partial y'} + \sin \theta \frac{\partial}{\partial z'}. \tag{25} \]

For a plane wave propagating along the \( y' \) axis, the electric field will be polarized along the \( x \) axis if it is an ordinary wave or it will, in general, have both \( y \) and \( z \) components if it is an extraordinary wave. In the spirit of the paraxial approximation we shall assume that the electric field corresponding to the beam is plane polarized with a polarization corresponding to either an ordinary or an extraordinary wave. This will be a good approximation so long as the directions of the component waves in the beam do not deviate much from the assumed direction of propagation.

With this plane-polarized model in mind, we can describe the propagation of an ordinary wave beam with Eq. (12a) by setting \( E_z = 0 \) and the propagation of an extraordinary-wave beam with Eqs. (12b) and (12c). For an ordinary-wave beam Eq. (12a) becomes
\[ \frac{\partial^2 E_x}{\partial x'^2} + \frac{\partial^2 E_x}{\partial y'^2} + n_e^2 k_o^2 E_x = 0, \tag{26} \]
or, in the rotated coordinate system,
\[ \frac{\partial^2 E_x}{\partial x'^2} + \frac{\partial^2 E_x}{\partial y'^2} + \frac{\partial^2 E_x}{\partial z'^2} + n_e^2 k_o^2 E_x = 0. \tag{27} \]

By expressing \( E_x(x, y', z') \) in the form
\[ E_x = \varepsilon(x, y', z') \exp(-i n_e k_o y'), \tag{28} \]
by substituting Eq. (28) into Eq. (27), and by neglecting \( \partial^2 \varepsilon/\partial y'^2 \), we obtain a standard paraxial wave equation
\[ 2i n_e k_o \frac{\partial \varepsilon}{\partial y'} = \frac{\partial^2 \varepsilon}{\partial x'^2} + \frac{\partial^2 \varepsilon}{\partial z'^2}, \tag{29} \]
which gives an accurate description of the beam as long as
\[ \frac{\partial^2 \varepsilon}{\partial y'^2} \ll k_o \frac{\partial \varepsilon}{\partial y'}. \tag{30} \]

This inequality implies, of course, that \( \varepsilon(x, y', z') \) does not vary much with \( y' \) over a wavelength distance.

Turning next to an extraordinary wave beam, we focus our attention first on Eq. (12c) for the \( z \) component. Writing this equation in the transformed coordinate system with the help of Eqs. (25), we obtain
\[ \frac{\partial^2 E_z}{\partial x'^2} + (\sin^2 \theta + \gamma^2 \cos^2 \theta) \frac{\partial^2 E_z}{\partial y'^2} + (\gamma^2 - 1) \sin 2 \theta \frac{\partial^2 E_z}{\partial y' \partial z'} + (\cos^2 \theta + \gamma^2 \sin^2 \theta) \frac{\partial^2 E_z}{\partial z'^2} + k_o n_e^2 E_z = 0. \tag{31} \]

For a plane-wave solution,
\[ E_z = \varepsilon_z (x, y', z') \exp(-i k_o y'), \tag{32} \]
Eq. (31) implies that
\[ (\sin^2 \theta + \gamma^2 \cos^2 \theta) k_o^2 = n_e^2 k_o^2. \tag{33} \]

Writing
\[ k_o^2 = n_e^2 k_o^2, \tag{34} \]
we obtain
\[ n^2 = \frac{n_e^2}{\sin^2 \theta + \gamma^2 \cos^2 \theta}, \tag{35} \]
from which one recovers relation (22), i.e.,
\[ \frac{1}{n^2} = \frac{n_e^2}{n_o^2} + \frac{\cos^2 \theta}{n_o^2}. \tag{22} \]

We now express \( E_z \) in the form
\[ E_z = \varepsilon_z (x, y', z') \exp(-i k_o y'), \tag{36} \]
and substitute into Eq. (31). Making use of Eq. (33), we find the following equation for \( \varepsilon_z \):

![Fig. 1. Coordinate systems and wave vector, which makes an angle \( \theta \) with the optic axis.](image-url)
\[
\frac{\partial^2 \mathbf{E}_z}{\partial x^2} + (\sin^2 \theta + \gamma^2 \cos^2 \theta) \left( \frac{\partial^2 \mathbf{E}_z}{\partial y'^2} - 2ik_y \frac{\partial \mathbf{E}_z}{\partial y'} \right) \\
+ (\gamma^2 - 1) \sin 2\theta \left( \frac{\partial^2 \mathbf{E}_z}{\partial y' \partial z'} - ik_y \frac{\partial \mathbf{E}_z}{\partial z'} \right) \\
+ (\cos^2 \theta + \gamma^2 \sin^2 \theta) \frac{\partial^2 \mathbf{E}_z}{\partial z'^2} = 0.
\]

To derive a paraxial wave equation we assume that
\[
\left| \frac{\partial \mathbf{E}_z}{\partial z'} \right| \ll k_y \left| \frac{\partial \mathbf{E}_z}{\partial y'} \right|,
\]
\[
\left| \frac{\partial^2 \mathbf{E}_z}{\partial y' \partial z'} \right| \ll k_y \left| \frac{\partial \mathbf{E}_z}{\partial z'} \right|.
\]

Making use of Eqs. (33)-(35) and neglecting second derivatives in accordance with Eqs. (38), we obtain the following paraxial wave equation:
\[
2i\pi \kappa_0 (\sin^2 \theta + \gamma^2 \cos^2 \theta)^{1/2} \left( \frac{\partial \mathbf{E}_z}{\partial y'} + \tan \theta \frac{\partial \mathbf{E}_z}{\partial z'} \right) = \frac{\partial^2 \mathbf{E}_z}{\partial x^2} \\
+ (\cos^2 \theta + \gamma^2 \sin^2 \theta) \frac{\partial^2 \mathbf{E}_z}{\partial z'^2}.
\]

Here
\[
\tan \theta' = \frac{\sin \theta \cos (\gamma^2 - 1)}{\gamma^2 \cos^2 \theta + \sin^2 \theta'}
\]

where \( \theta' \) is the angle between the Poynting vector \( \mathbf{S} \) and the direction of the wave vector \( \mathbf{k} \) (see Fig. 2). This relationship is established in Appendix A. If we introduce the variable
\[
z'' = z' - \tan \theta' y',
\]
Eq. (39) becomes
\[
2i\pi \kappa_0 (\sin^2 \theta + \gamma^2 \cos^2 \theta)^{1/2} \frac{\partial \mathbf{E}_z}{\partial y'} \frac{\partial \mathbf{E}_z}{\partial x^2} \\
+ (\cos^2 \theta + \gamma^2 \sin^2 \theta) \frac{\partial^2 \mathbf{E}_z}{\partial z'^2} = 0.
\]

To summarize, the \( z \) component of the electric field is described by a paraxial wave equation in which the effective wave number is
\[
k_{\text{eff}} = n_\kappa \kappa_0 (\sin^2 \theta + \gamma^2 \cos^2 \theta)^{1/2} \\
= k_\kappa n_\kappa^2 / n
\]

and the coordinate in the principal plane is rescaled by the factor \((\cos^2 \theta + \gamma^2 \sin^2 \theta)^{-1/2}\). At the same time, the field is displaced in a direction transverse to the direction of the wave vector by an amount
\[
\Delta z = y' \tan \theta'.
\]

This represents an offset of the extraordinary-wave beam relative to the ordinary-wave beam, whose propagation characteristics are described by Eq. (29), a phenomenon that is well known to anyone who has observed the double image that results when a dot is viewed through a calcite crystal. When the wave propagates in a direction normal to the \( z \) axis, the offset disappears and Eq. (42) becomes
\[
2i\pi \kappa_0 \frac{\partial \mathbf{E}_z}{\partial z'} = \frac{\partial^2 \mathbf{E}_z}{\partial x^2} + n_{\kappa}^2 \frac{\partial^2 \mathbf{E}_z}{\partial z'^2}.
\]

As \( \theta \) approaches zero, on the other hand, the whole formulation breaks down, since the extraordinary characteristic wave solution cannot exist for \( \theta = 0 \).

Finally, we can express the \( y \) component of the field in the form
\[
\mathbf{E}_y = \mathbf{E}_y(x, y', z') \exp(-ik_y y),
\]

where \( \mathbf{E}_y(x, y', z') \) is slowly varying in comparison with the exponential factor and \( k_y \) is defined by Eqs. (33)-(35). Form (46) is dictated by the fact that we are seeking solutions composed of superposed plane waves that do not depart greatly from the corresponding characteristic plane-wave solution. Writing Eq. (9) in the rotated coordinate system and making use of Eq. (46), we have
\[
-\epsilon_z \left[ \sin \theta \left( ik_y \mathbf{E}_y - \frac{\partial \mathbf{E}_z}{\partial y'} \right) + \cos \theta \frac{\partial \mathbf{E}_z}{\partial z'} \right] \\
-\epsilon_z \left[ \cos \theta \left( ik_y \mathbf{E}_y - \frac{\partial \mathbf{E}_z}{\partial y'} \right) - \sin \theta \frac{\partial \mathbf{E}_z}{\partial z'} \right] = 0.
\]

For a paraxial beam the derivatives in Eq. (47) can be neglected in comparison with \( k_y \mathbf{E}_y \) and \( k_y \mathbf{E}_z \), except for \( \theta \) close to 0. In this approximation we have
\[
\mathbf{E}_y = -\frac{\epsilon_z}{\epsilon_x} \cot \theta \mathbf{E}_z.
\]

Since \( \mathbf{E}_y \) is proportional to \( \mathbf{E}_z \), the field for the extraordinary beam is completely specified when Eq. (42) is solved for \( \mathbf{E}_z \).

5. METHOD OF SOLUTION AND APPLICATION TO PROPAGATING GAUSSIAN BEAMS AND DIFFRACTION FROM A CIRCULAR APERTURE

Equation (42) can be solved in terms of Fourier transforms, which in turn can be evaluated accurately by means of the fast Fourier transform (FFT) algorithm. If \( \mathbf{E}_z \) is expressed as
\[
\mathbf{E}_z(x, y', z') = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_z \mathbf{E}_z(k_x, y', k_z) \\
\times \exp[-i(k_x x + k_z z')],
\]

\( \mathbf{E}_z(k_x, y', k_z) \) satisfies
\[
\mathbf{E}_z(k_x, y', k_z) = \mathbf{E}_z(k_x, 0, k_z) \\
\times \exp \left\{ -i \left[ k_x^2 + k_z^2 (\cos^2 \theta + \gamma^2 \sin^2 \theta) \right] / 2k_\kappa n_\kappa^2 / n \right\} y'.
\]
The method for evaluating Eq. (49) in terms of FFT's has already been discussed at length. If the initial field is the Gaussian function
\[ G(x,0)G(z', 0), \]
direct substitution into Eqs. (49) and (50) gives
\[ G(x, y') = \frac{\sigma}{\sqrt{\pi}} e^{-\frac{x^2}{\sigma^2}} e^{-i\frac{ny'/\sigma n_e^2 n_0}{2}}, \]
where
\[ G(x, y') = \frac{\sigma}{\sqrt{\pi}} e^{-\frac{x^2}{\sigma^2}} \exp \left\{ -i\frac{ny'/\sigma n_e^2 n_0}{2} \right\}, \]
From Eqs. (52) and (53) it is evident that the beam spreads differently in the x and z' directions and that the beam phase front is elliptic rather than circular as it would be propagating in an isotropic medium.

Figures 3–7 deal with extraordinary wave intensity patterns that would result from diffraction in calcite by a uniformly illuminated circular aperture for \( \lambda = 5893 \) Å. Computations were made with Eqs. (49) and (50) and \( n_e = 1.658 \) and \( n_0 = 1.486 \).

Figures 3–5 pertain to a Fresnel number \( N = a^2/\lambda y' = 5 \), where \( a \) is the aperture radius. Figure 3 shows the intensity as a function of both \( x \) and \( z' \). Figure 3(a), which is displayed for reference, shows the intensity computed for the standard paraxial equation with \( n = n_e \). Figures 3(b) and 3(c) show the intensity patterns computed with \( \theta = 45^\circ \) and \( \theta = 90^\circ \), respectively, for the correct paraxial Eq. (42). Figures 4 and 5, which show the iso-intensity contours for \( \theta = 45^\circ \) and \( \theta = 90^\circ \), indicate the degree to which the circular symmetry of the diffraction pattern has been disrupted by the birefringence of the medium.

Figures 6 and 7, which show results for \( N = 0.1 \), indicate the behavior of the far-field diffraction pattern. Figure 6(a) shows the reference intensity patterns computed for the standard paraxial equation with \( n = n_e \), and Figs. 6(b) and 6(c) show the correct patterns for \( \theta = 45^\circ \) and \( \theta = 90^\circ \), re-
spectively. The central diffraction peak is seen to be higher and narrower in the presence of birefringence. This is a consequence of a compression of the pattern along the z' direction that is evident in the isointensity contours in Fig. 7.

6. EXTENSION TO WEAKLY GUIDING MEDIA

If \( \epsilon_x \) and \( \epsilon_z \) are allowed to vary with position, Eq. (10) is replaced by

\[
\nabla \cdot \mathbf{E} = -E_x \left( \frac{1}{\epsilon_x} \frac{\partial \epsilon_x}{\partial x} \right) - E_y \left( \frac{1}{\epsilon_y} \frac{\partial \epsilon_y}{\partial y} \right) - E_z \left( \frac{1}{\epsilon_z} \frac{\partial \epsilon_z}{\partial z} \right) + \left( 1 - \frac{\epsilon_x}{\epsilon_z} \right) \frac{\partial E_z}{\partial z},
\]

(54)

Let us put

\[
\epsilon_x = \epsilon_x^0 + \Delta \epsilon_x,
\]

\[
\epsilon_z = \epsilon_z^0 + \Delta \epsilon_z.
\]

(55)

If \( \Delta \epsilon_x/\epsilon_x^0 \) and \( \Delta \epsilon_z/\epsilon_z^0 \) are sufficiently small, the first three right-hand terms of Eq. (54) can be neglected. If the remaining term is not everywhere negligible, one can write

\[
\nabla \cdot \mathbf{E} \approx (1 - \gamma^2) \frac{\partial E_z}{\partial z},
\]

(56)

where

\[
\gamma^2 = \frac{\epsilon_x^0}{\epsilon_z^0}.
\]

(57)

It can then be assumed that Eqs. (12a)-(12c) describe the field with variable \( \epsilon_x \) and \( \epsilon_z \).

If we write further that

\[
n_0 = n_0^0 + \Delta n_0,
\]

\[
n_\epsilon = n_\epsilon^0 + \Delta n_\epsilon,
\]

(58)

Eq. (29) will be replaced by

\[
2n_0 k_0 \frac{\partial \epsilon_x}{\partial y'} = \frac{\partial^2 \epsilon_x}{\partial x'^2} + \frac{\partial^2 \epsilon_x}{\partial z'^2} + 2\Delta n_0 k_0 \epsilon_x^0 \epsilon_x
\]

(59)

and Eq. (42) will be replaced by

\[
2n_\epsilon k (\sin^2 \theta + \gamma^2 \cos^2 \theta)^{1/2} \frac{\partial \epsilon_x}{\partial y'} = \frac{\partial^2 \epsilon_x}{\partial x'^2} + \frac{\partial^2 \epsilon_x}{\partial z'^2} + (\cos^2 \theta + \gamma^2 \sin^2 \theta) \frac{\partial^2 \epsilon_x}{\partial z'^2} + 2\Delta n_\epsilon k_0 \epsilon_x^0 \epsilon_x,
\]

(60)

where terms of order \( (\Delta n_0)^2 \) and \( (\Delta n_\epsilon)^2 \) have been neglected.

Equations (59) and (60) can be applied to weakly guiding optical waveguides formed in birefringent substrates, provided that \( n_0^0 \) and \( n_\epsilon^0 \) do not vary with position. The spectral methods outlined in Refs. 9 and 10 can be used to determine all waveguide properties, such as mode-propagation constants and eigenfunctions.

7. SUMMARY AND CONCLUSION

From Maxwell’s equations we have derived a set of generalized wave equations that take the place of the vector Helmholtz equation in a uniform uniaxial anisotropic medium. These equations permit a unified treatment of both plane waves and beams. From the generalized wave equations we have derived paraxial wave equations valid for an arbitrary direction of propagation. An ordinary-wave beam is described by a standard paraxial equation. An extraordinary-wave beam, on the other hand, is described by a paraxial equation that
introduces both a transverse shift in the beam position and a rescaling of the transverse coordinate in the principal plane. These equations can be generalized to nonuniform media with weak variations in refractive index. The numerical solution of the paraxial equations can be carried out in terms of FFT’s using standard techniques.

APPENDIX A. RELATING BEAM DISPLACEMENT TO ANGLE BETWEEN POYNTING AND WAVE VECTORS FOR AN EXTRAORDINARY-WAVE BEAM

In Section 4 we derived an equation governing the propagation of an extraordinary-wave beam, which introduces a displacement of the beam from a propagation axis directed along the wave vector. We now show that the beam is displaced as though it propagated in the direction of the Poynting vector.

The wave and Poynting vectors for a paraxial beam are those of the carrier wave. Writing

\[ E = \xi^0 \exp[-i(k_y y + k_z z)], \]

we obtain, from Eq. (7),

\[ \xi_z^0 = -\xi_z \tan \theta \xi_y^0. \]

Consequently, the unit vector \( \hat{e} \) in the direction of the electric field is

\[ \hat{e} = \left( 0, \frac{1}{1 + \left( \frac{\xi_x}{\xi_z} \tan \theta \right)^{1/2}}, -\frac{\xi_x}{\xi_z} \tan \theta \right). \]

From Eq. (1b), the unit vector \( \hat{h} \) in the direction of the magnetic field is

\[ \hat{h} = (1, 0, 0), \]

and the unit vector \( \hat{s} \) in the direction of the Poynting vector is

\[ \hat{s} = \hat{e} \times \hat{h} = \left( 0, \frac{\xi_x}{\xi_z} \tan \theta, -1 \right) \left( 1 + \left( \frac{\xi_x}{\xi_z} \tan \theta \right)^{1/2}, 0, 1 + \left( \frac{\xi_x}{\xi_z} \tan \theta \right)^{1/2} \right)^{-1}. \]

The unit vector \( \hat{k} \) in the direction of the wave vector is

\[ \hat{k} = (0, \sin \theta, \cos \theta). \]

Consequently,

\[ \cos \theta' = \hat{k} \cdot \hat{s} = -\frac{\xi_x}{\xi_z} \tan \theta \sin \theta - \cos \theta \]

\[ = -\frac{\xi_x}{\xi_z} \tan \theta \sin \theta - \cos \theta \]

\[ = \frac{\xi_x}{\xi_z} \tan \theta \sin \theta - \cos \theta \]

\[ = \frac{\xi_x}{\xi_z} \tan \theta \sin \theta - \cos \theta \]

\[ = \frac{1}{\xi_z} \left( \frac{\xi_x}{\xi_z} \tan \theta \sin \theta - \cos \theta \right) \]

and

\[ \tan \theta' = (\sec^2 \theta' - 1)^{1/2} \]

\[ = \frac{\xi_x}{\xi_z} \cos \theta + \sin^2 \theta \]

which is Eq. (40). The displacement [Eq. (40)] thus moves the beam in the direction \( \hat{s} \).

ACKNOWLEDGMENT

This research was performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore National Laboratory under contract no. W-7405-ENG-48.

REFERENCES

2. For a comprehensive treatment of the propagation of Gaussian beams, see, for example, J. A. Arnaud, Beam and Fiber Optics (Academic, New York, 1976), Chap. 2.
9. This result could be anticipated from the conventional treatment of birefringence. See, for example, Ref. 1. However, the fact that the offset is automatically incorporated in the paraxial wave equation gives assurance that it is correct.